2012 Dirac medal ceremony, ICTP Trieste, Italy, July 4, 2013.

Topologically-non-trivial phases in condensed matter physics: from 1D spin chains, to 2D Chern insulators, to today

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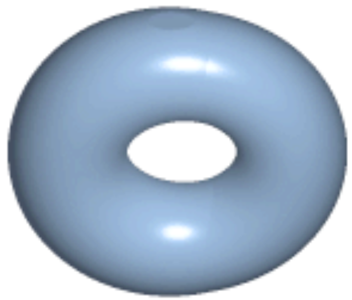
- Topology and quantum physics
- Spin Chains and Chern Insulators as Symmetry Protected Topological states
- Lessons for the future

Topological equivalence

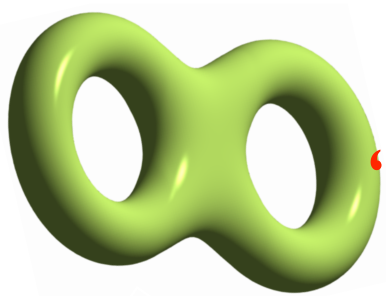
genus 0

mug



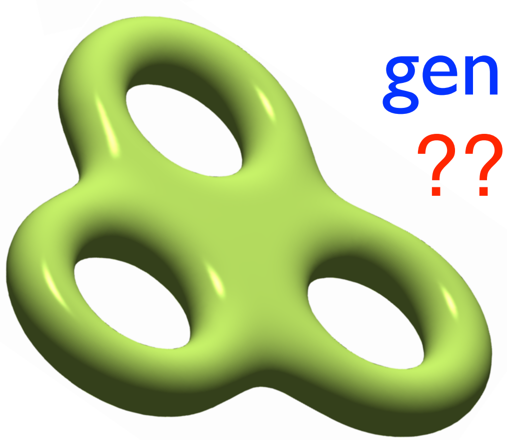
genus 1

coffee cup



genus 2

“loving cup” (or soup?)



genus 3

???

- inequivalent objects cannot be continuously transformed into each other
- these 2D surfaces are classified by an integer, their *genus* (number of handles)

topology (genus) is a **global** property

- **geometric** properties (such as curvature) are ***local*** properties
- but integrals over local geometric properties may characterize global topology!

Gauss-Bonnet (for a closed surface)

$$\int d^2\boldsymbol{r}(\text{Gaussian curvature}) = 4\pi(1-\text{genus})$$

$$= 2\pi(\text{Euler characteristic})$$

(Gaussian curvature $= \frac{1}{R_1 R_2}$)

$$4\pi r^2 \times \frac{1}{r^2} = \qquad 4\pi(1-0)$$

- trivially true for a sphere, but non-trivially true for any compact 2D manifold

- In quantum mechanics, “geometry” relates to energy, “local deformations” become adiabatic changes of the Hamiltonian, and “smoothness” (short-distance regularization) of the manifold derives from an energy gap

- the topology of quantum states is conserved so long as energy gaps do not close.

- A more abstract generalization of the Gauss-Bonnet formula due to Chern found its way into quantum condensed-matter physics in the 1980's

- Quantum states are **ambiguous** up to a phase:
- Physical properties are defined by expectation values $\langle\Psi|\hat{O}|\Psi\rangle$ that are left unchanged by

$$|\Psi\rangle \mapsto e^{i\varphi}|\Psi\rangle$$

- As noticed by Berry, this has profound consequences for a family of quantum states parametrized by a continuous d-dimensional coordinate x *in a parameter space.*

- $|\Psi(\boldsymbol{x})\rangle$ can be expanded in a fixed orthonormal basis

$$|\Psi(\boldsymbol{x})\rangle = \sum_i u_i(\boldsymbol{x})|i\rangle \qquad \langle i|j\rangle = \delta_{ij}$$

$$|\partial_\mu \Psi(\boldsymbol{x})\rangle \equiv \sum_i \frac{\partial u_i(\boldsymbol{x})}{\partial x^\mu}|i\rangle$$

- the simple derivative $|\partial_\mu \Psi(\boldsymbol{x})\rangle$ does not transform "nicely" under $|\Psi\rangle \mapsto e^{i\varphi}|\Psi\rangle$
- we need a "gauge-covariant" derivative

$$|D_\mu \Psi(\boldsymbol{x})\rangle = |\partial_\mu \Psi(\boldsymbol{x})\rangle - |\Psi(\boldsymbol{x})\rangle\langle\Psi(\boldsymbol{x})|\partial_\mu \Psi(\boldsymbol{x})\rangle$$

projects out parts of $|\partial_\mu \Psi(\boldsymbol{x})\rangle$ not orthogonal to $|\Psi(\boldsymbol{x})\rangle$

$$\langle\Psi(\boldsymbol{x})|D_\mu \Psi(\boldsymbol{x})\rangle = 0$$

parallel transport

- The gauge-covariant derivative can also be written

$$|D_\mu \Psi(\boldsymbol{x})\rangle = |\partial_\mu \Psi(x)\rangle - i\mathcal{A}_\mu(\boldsymbol{x})|\Psi(\boldsymbol{x})\rangle$$

Lots of analogies with electromagnetic gauge fields in Euclidean space!

an analog of the electromagnetic vector potential in the parameter space x

- Berry's phase factor for a closed path Γ in parameter space is the analog of a Bohm-Aharonov phase

$$e^{i\phi_\Gamma} = \exp i \oint_\Gamma dx^\mu \mathcal{A}_\mu(\boldsymbol{x})$$

- The key gauge-invariant quantity is

$$\langle D_\mu \Psi(\boldsymbol{x}) | D_\nu \Psi(\boldsymbol{x})\rangle = \tfrac{1}{2}\left(\mathcal{G}_{\mu\nu}(x) + i\mathcal{F}_{\mu\nu}(\boldsymbol{x})\right)$$

Real symmetric positive Fubini-study metric (defines "quantum geometry")

Real antisymmetric Berry curvature

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$$

Chern's generalization of Gauss-Bonnet

$$\int_{\mathcal{M}_2} dx^\mu \wedge dx^\nu \mathcal{F}_{\mu\nu}(\boldsymbol{x}) = 2\pi \mathbb{C}_1$$

integral over a closed orientable 2-manifold

"Chern number"

first Chern class (an integer) replaces Euler's characteristic

- The simplest example of this is the quantum geometry of the coherent states of a quantum spin

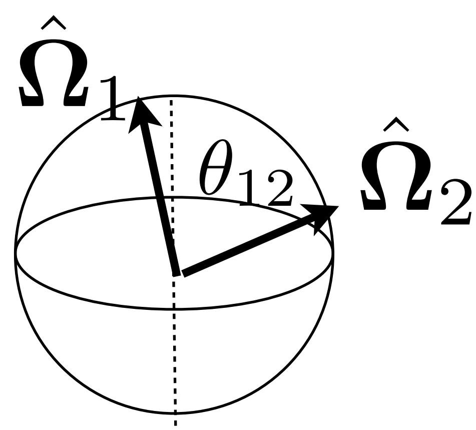
"most classical state of a spin"

$$\hat{\mathbf{\Omega}} \cdot \boldsymbol{S}|\hat{\mathbf{\Omega}}\rangle = S|\hat{\mathbf{\Omega}}\rangle \qquad \hat{\Omega} \cdot \hat{\Omega} = 1$$

$$|\hat{\mathbf{\Omega}} \times \boldsymbol{S}|^2|\hat{\mathbf{\Omega}}\rangle = S|\hat{\mathbf{\Omega}}\rangle \qquad \boldsymbol{S} \cdot \boldsymbol{S} = S(S+1)$$

spin has maximum polarization along direction $\hat{\mathbf{\Omega}}$ but still has zero-point motion around it

$$|\langle \hat{\mathbf{\Omega}}_1 | \hat{\mathbf{\Omega}}_2 \rangle| = (\cos \tfrac{1}{2}\theta_{12})^{2S}$$



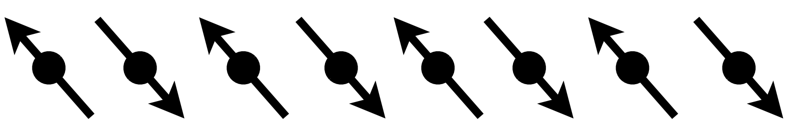
coherent states are non-orthogonal and overcomplete!

- This “explains” a curious feature of the Heisenberg exchange Hamiltonian for spins (and also “explains” very topical things such as Laughlin states in fractional Chern insulators...) I hope to cover this in my talk tomorrow

$$H = J \sum_{\langle i,j \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j$$

Coherent state of ordered antiferromagnet

$$\hat{\boldsymbol{\Omega}} \cdot \boldsymbol{S}_i |\hat{\boldsymbol{\Omega}}_{\text{Néel}}\rangle = (-1)^i S |\hat{\boldsymbol{\Omega}}_{\text{Néel}}\rangle$$

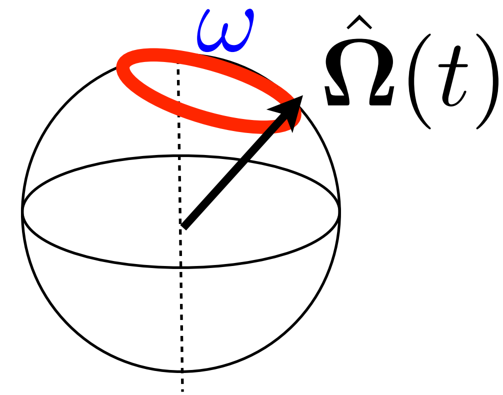


- The Hamiltonian has <u>**NO KINETIC ENERGY**</u>!

- For large S (or large second-neighbor ferromagnetic exchange) the Néel state has a very low energy, and quantum dynamics arises only because the coherent states are NON-ORTHOGONAL with a non-trivial Fubini-Study metric (quantum geometry)

- Before the 1980's, condensed matter theorists rarely used Lagrangians or actions
- The spin S was assumed to act like a continuously variable real parameter in the Heisenberg Hamiltonian, merely controlling the amount of local zero-point fluctuation about the classically-ordered state
- But, in one-spatial dimension, quantum fluctuations destroy long-range order, so GLOBAL (topological) structure becomes important.

- Dirac's 1931 quantization of the magnetic monopole also explains the topological quantization of spin so 2S is an integer!



$$e^{S/\hbar} = \int D\hat{\mathbf{\Omega}}(t) e^{iS\omega[\hat{\mathbf{\Omega}}(t)]} e^{-i\hbar^{-1}\oint dt\, H_S(\hat{\mathbf{\Omega}})}$$

topologically quantized: 2S = integer to make "Dirac string" invisible

here S can vary continuously

functional integral over histories $\hat{\mathbf{\Omega}}(t)$

$$\omega[\hat{\mathbf{\Omega}}(t)] = \int dt A_i(\hat{\mathbf{\Omega}}(t))\partial_t \hat{\Omega}^i(t)$$

solid angle swept out by spin history (only defined modulo 4π)

Dirac monopole vector potential

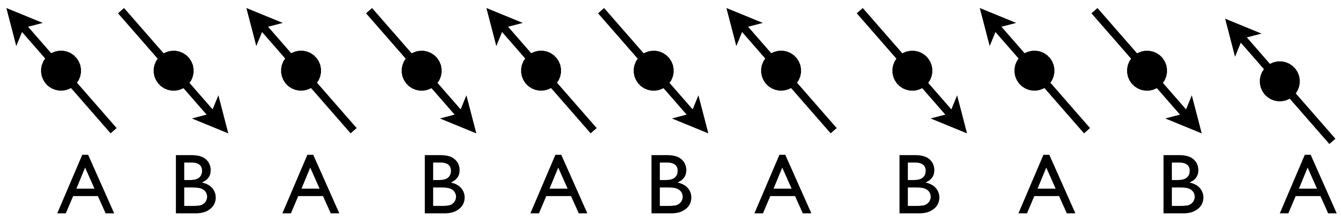
linear in time derivative, absent in Hamiltonian!

- The coherent state of the spin "lives" on a 2-manifold which is the unit sphere.
- Its Chern number is 2S, which must be topologically-quantized to be an integer*
- There is a further "Z_2" classification by whether 2S is even or odd (integer or half-odd-integer spins).

* the topological nature of this result is hidden in the standard algebraic derivation!

1D Quantum Heisenberg antiferromagnets (with time-reversal and inversion symmetry)

classical picture of (locally-) ordered state



- Zero-point quantum fluctuations in 1D destroy long-range order in the ground state

- effective field-theory = "non-linear sigma model"

FDMH 1983

$$\vec{\boldsymbol{S}}_n = (1)^n S\hat{\boldsymbol{\Omega}}(x_n)$$

unit-vector field $\hat{\boldsymbol{\Omega}}(x,t)$

- effective action

spin-wave velocity

$$\mathcal{S} = \frac{1}{2g}\int dxdt\left(c^{-1}|\partial_t\hat{\boldsymbol{\Omega}}|^2 - c|\partial_x\hat{\boldsymbol{\Omega}}|^2\right)$$
$$+\frac{\theta}{4\pi}\int dxdt\left(\hat{\boldsymbol{\Omega}}\cdot\partial_t\hat{\boldsymbol{\Omega}}\times\partial_x\hat{\boldsymbol{\Omega}}\right)$$

surface area on unit sphere swept out by space-time path of field $\hat{\boldsymbol{\Omega}}(x,t)$

if inversion or time-reversal symmetry is present: $e^{i\theta} = e^{-i\theta}$

$$\theta = 2\pi S$$

Topological "theta" term

- can be derived from Berry phases of individual spins
- Linear in time derivatives, so absent from Hamiltonian, does not affect classical-limit equation of motion.

- quantum ground state:

half-integer spin: $e^{i\theta} = -1$

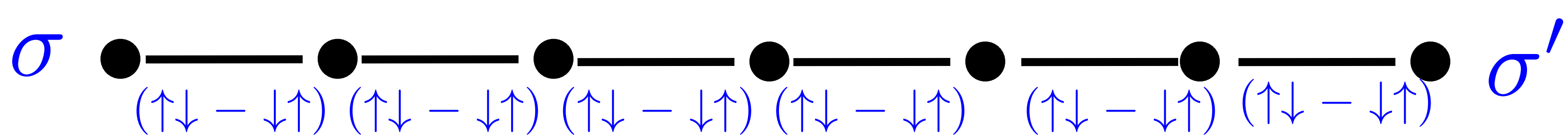
either gapless critical state (small g)

or gapped state with broken inversion symmetry

integer spin: $e^{i\theta} = +1$

gapped (incompressible) state,unbroken symmetry

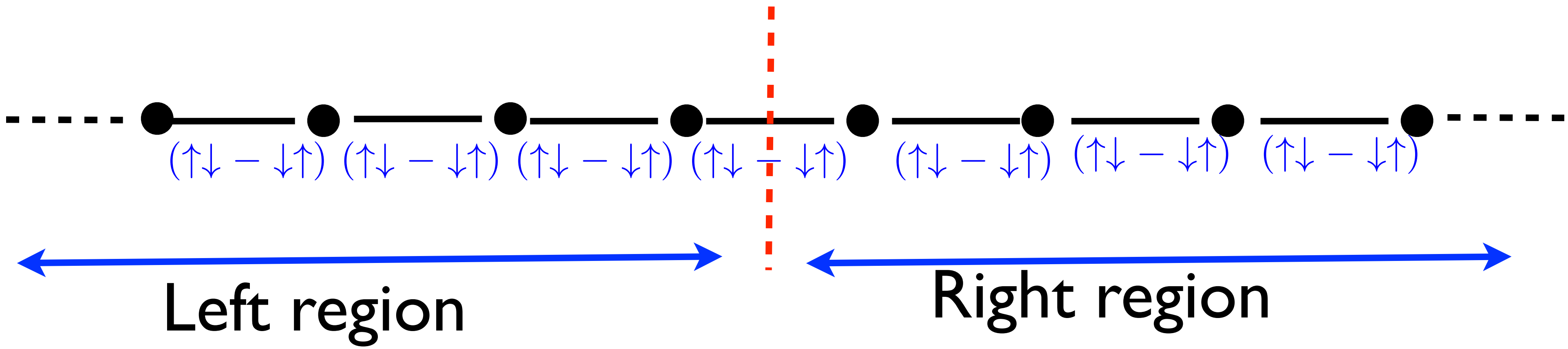
free spin-(S/2) states at free ends!



valence bond picture (AKLT) spin -1

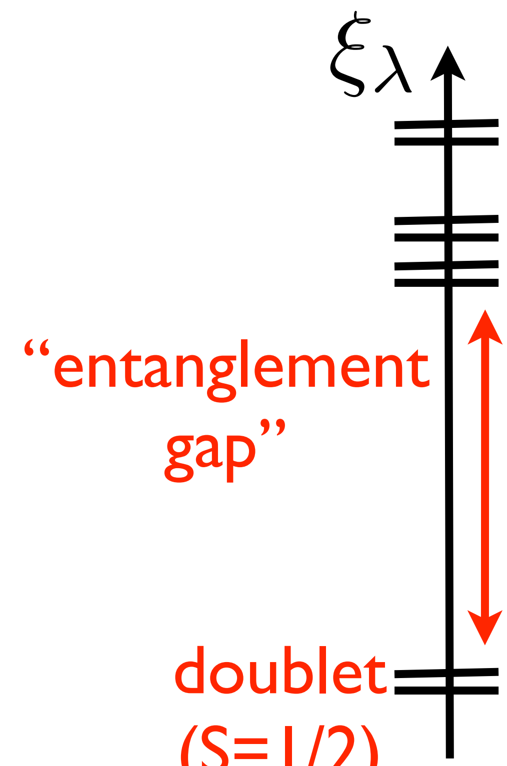
"symmetry-protected" topological order (Wen)

- topological order = long-range entanglement



$$|\Psi\rangle = \sum_\lambda e^{-\xi_\lambda/2} |\Psi_\lambda^L\rangle \otimes |\Psi_\lambda^R\rangle$$

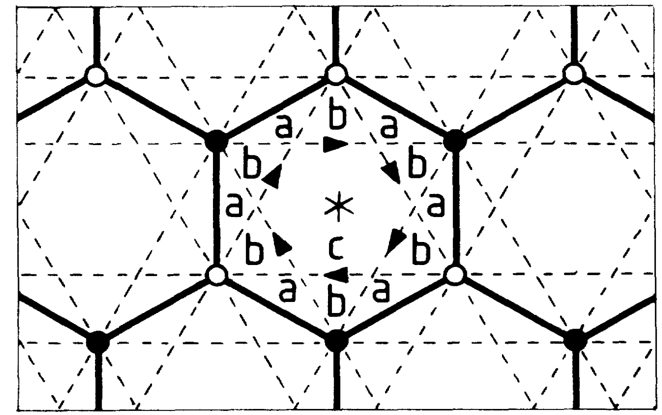
Bipartite Schmidt-decomposition of ground state reveals entanglement



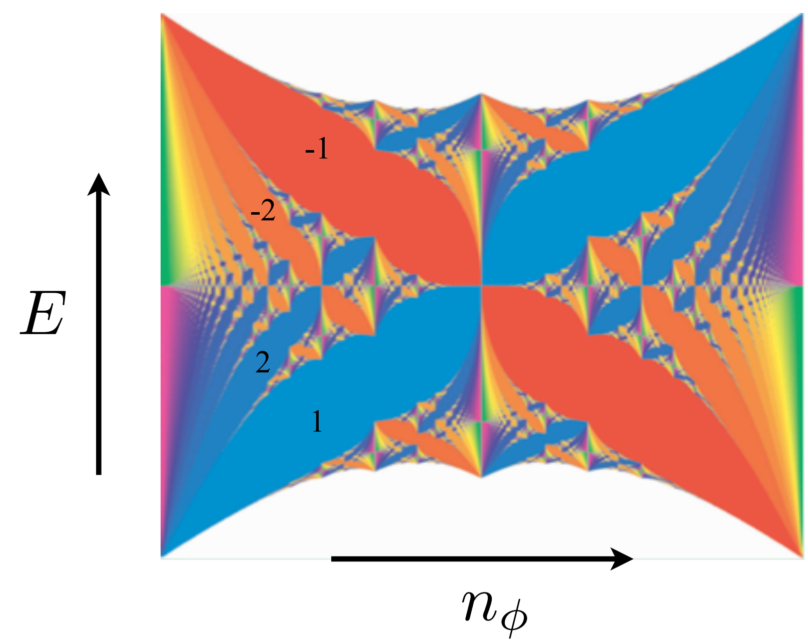
- a gapless "topological entanglement spectrum" separated from other Schmidt eigenvalues by an "entanglement gap" is characteristic of long-range topological order (Li + FDMH, PRL 2008)

The 2D Chern insulator as the first "topological insulator"

- This was a model for a "quantum Hall effect without Landau levels" (FDMH 1988), now variously known as the "quantum anomalous Hall effect" or "Chern insulator".



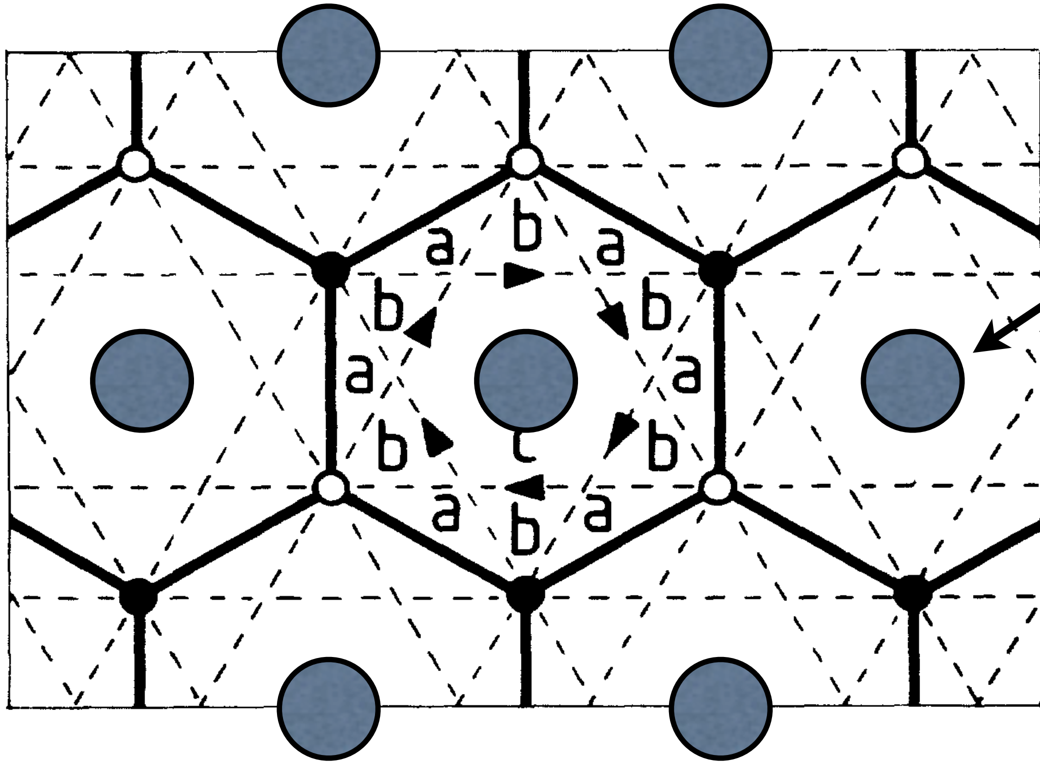
- Previously, Thouless, Kohmoto, Nightingale and den Nijs (TKNN) had analysed the QHE in the Hofstadter model, and found the invariant subsequently identified by Simon as the Chern number.



colored by Avron et al.

- In the Hofstadter model, a magnetic flux of p/q London (h/e) quanta passes through each unit cell of a periodic lattice, and reduces the Brillouin zone to a reciprocal-space area that is $1/q^2$ smaller, and each of these "magnetic bands" has an internal q-fold degeneracy. (p and q are relative primes). **For q=1, the Hofstadter model has a standard non-topological bandstructure**

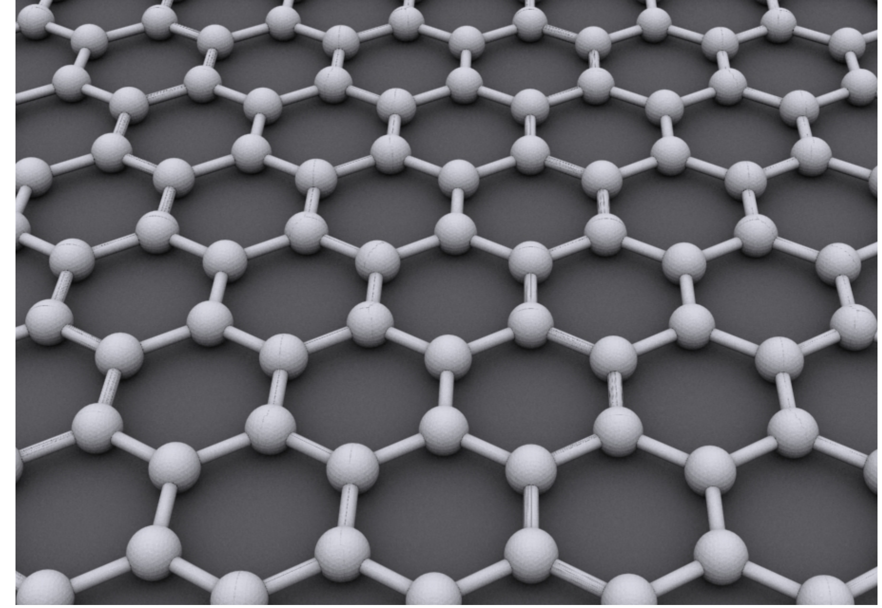
    - The 1988 modified graphene model I proposed seems to be the first that explicitly considered a system with $q = 1$ (and p = 0) where a Bloch band-structure that was "normal" (apart from broken time-reversal symmetry) produced a quantum Hall effect



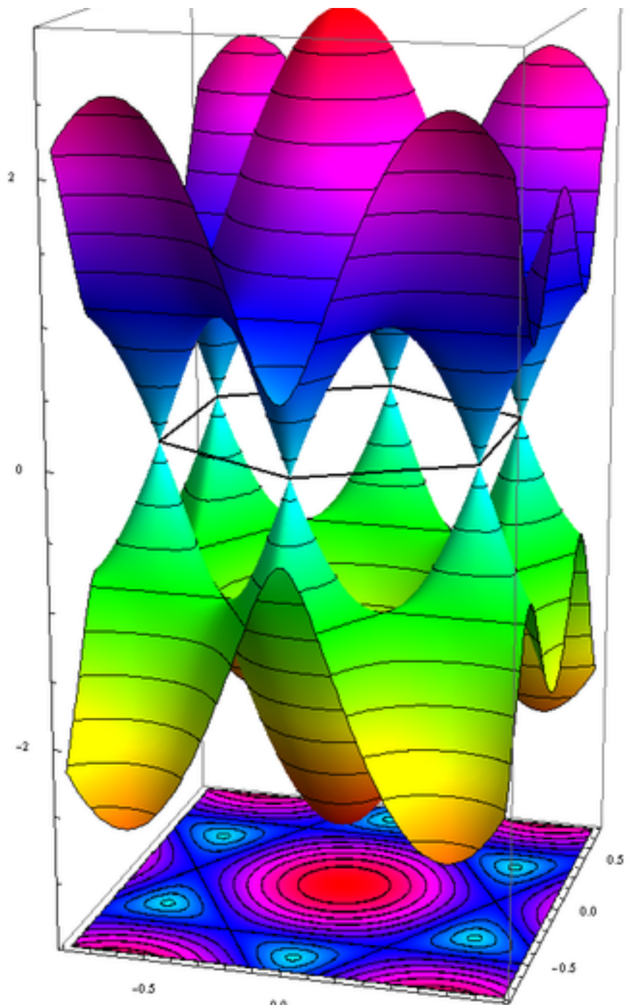
Ferromagnetically-ordered
magnetic dipoles
pointing out of the plane

- instead of an external magnetic field, the "toy model" placed a magnetic dipole at the center of each hexagon, which both spin-polarizes the electrons, and gives the same chiral phase to any triangular second-neighbor hopping path around a dipole.
- six-fold rotation symmetry is unbroken,
- no net flux through the unit cell

2D Graphene:



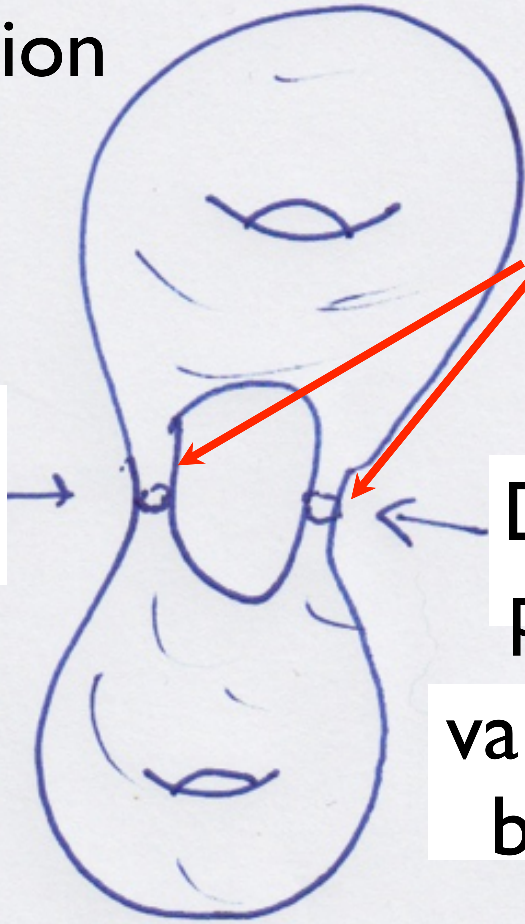
- Dirac points (2 valleys)



(looks like six points, but only two "Dirac points" are distinct)

Hilbert-Space picture of the geometry of the graphene valence and conduction band

(Brillouin zone of an isolated 2D band is a simple torus with genus 1)



In k-space, Dirac points are singularities, in Hilbert space, they are smooth tubes or "wormholes" that join the two bands

The two bands joined at the Dirac points form a Genus-3 manifold

- The underlying graphene band-structure with bands touching at two conjugate "Dirac points" at the corners of the hexagonal Brillouin zone requires vanishing Berry curvature $\mathcal{F}^{xy}(\boldsymbol{k})$ of the Bloch bands in the Brillouin zone (topologically a 2-torus).

  - spatial inversion symmetry implies

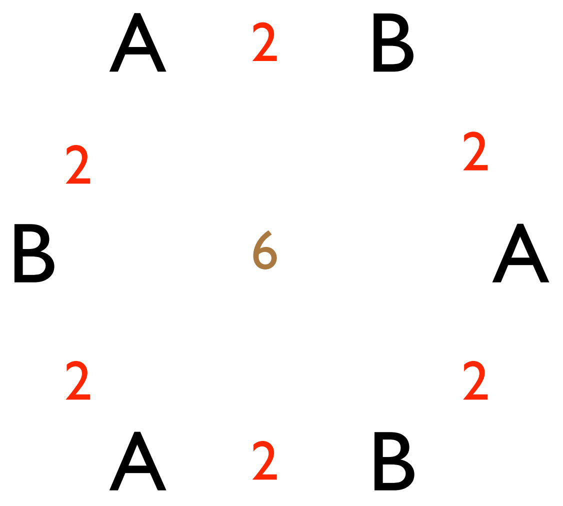
    $$\mathcal{F}^{xy}(\boldsymbol{k}) = \mathcal{F}^{xy}(-\boldsymbol{k})$$

  - time-reversal symmetry implies

    $$\mathcal{F}^{xy}(\boldsymbol{k}) = -\mathcal{F}^{xy}(-\boldsymbol{k})$$

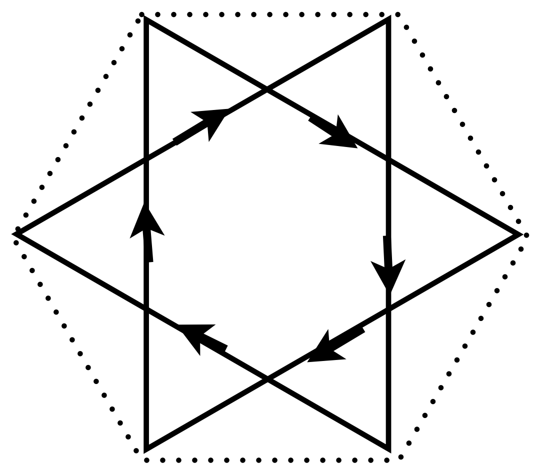
- <u>**Both**</u> symmetries are required for the Dirac band-touching points to survive

- there are two sites per unit cell, related by 2D inversion (180° rotation) symmetry



- breaking this symmetry opens a "boring" equal-sign mass gap at both Dirac points, and the system becomes a simple "Boron-nitride" semiconductor

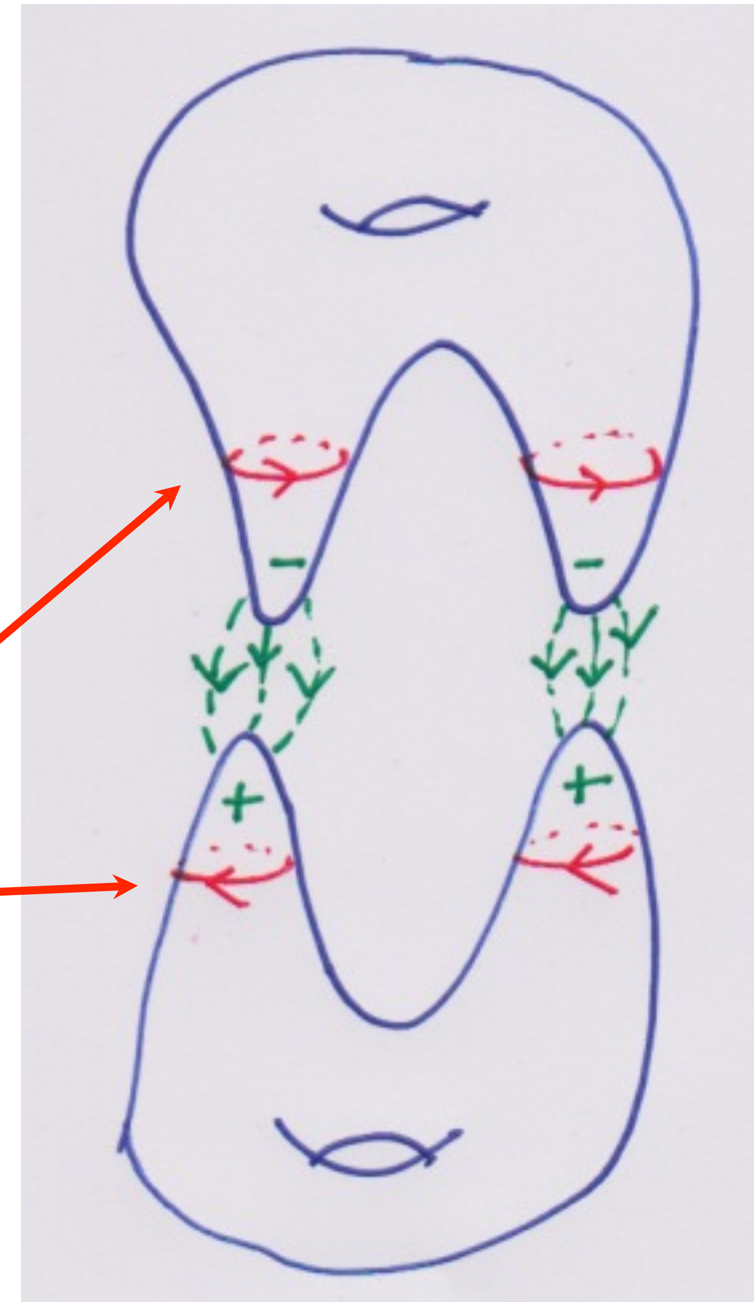
- In contrast, breaking time-reversal gives an "interesting" mass term that has opposite sign at the two Dirac points



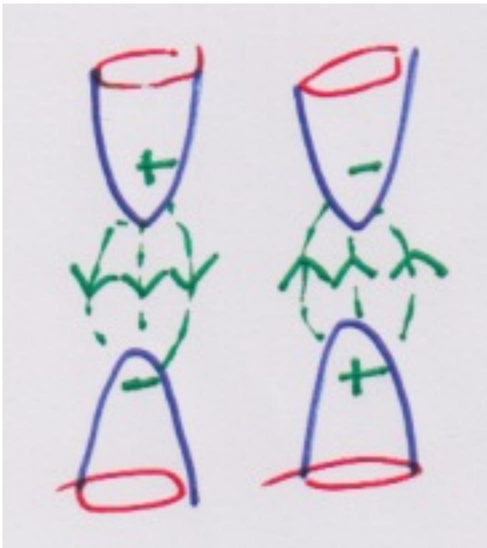
- This "topological" mass term anticommutes with **all** other possible mass terms (which commute with each other) and gives a quantum Hall effect.

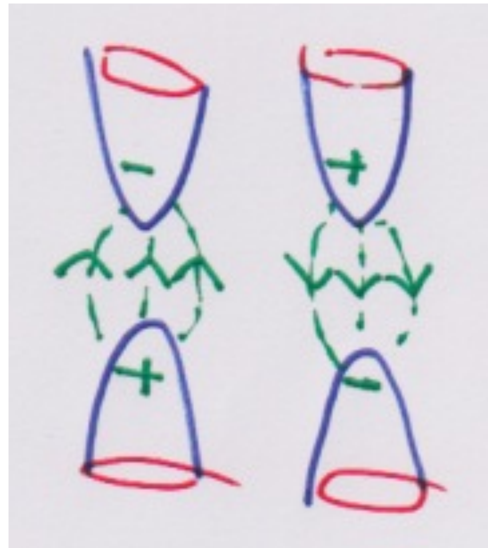
- Breaking either Inversion or time-reversal allows a local Berry-curvature at each point in the Hilbert space.
- The Dirac points lose topological protection and split

Berry "flux" of close to π flows through these paths. It must emerge or enter near the former Dirac points

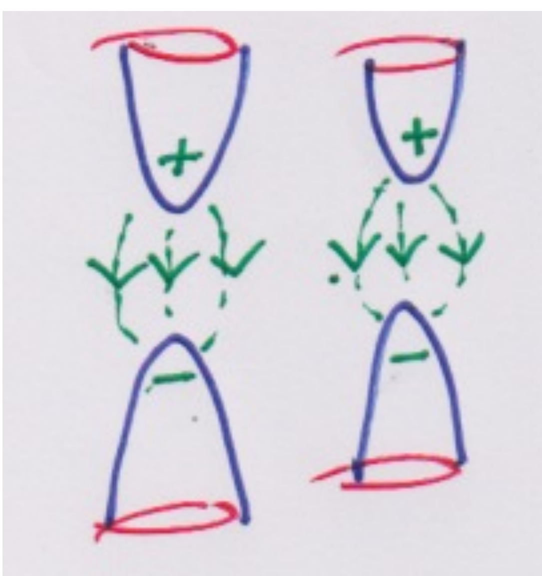


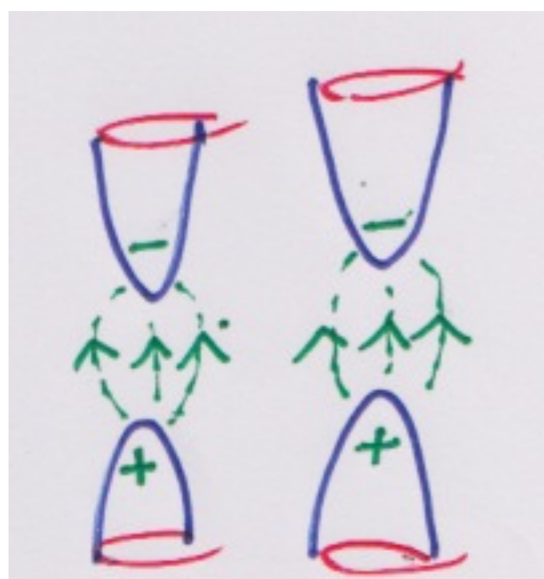
four possibilities





Broken inversion symmetry, no net Berry flux flows between bands





Broken time reversal symmetry, net Berry flux 2π flows between bands. bands have Chern numbers $+1$, -1

- If both perturbations are present, the Dirac points have different mass gaps

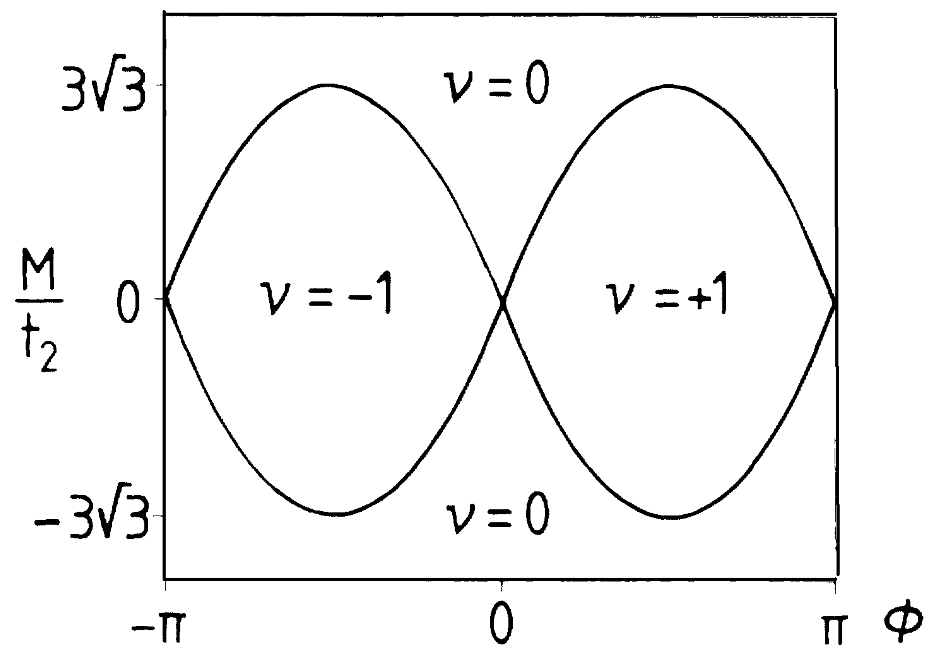
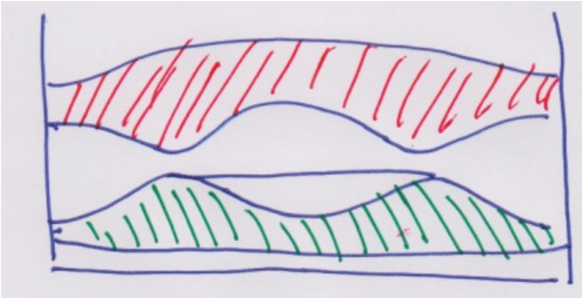
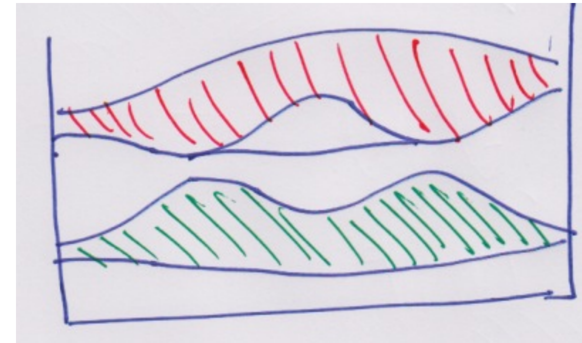


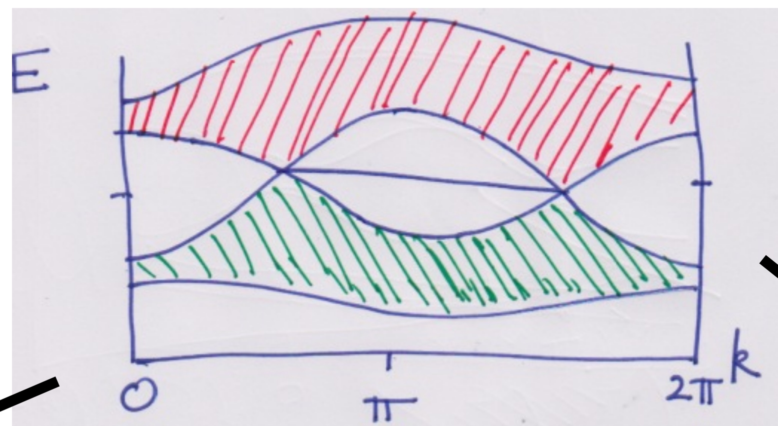
FIG. 2. Phase diagram of the spinless electron model with $|t_2/t_1| < \frac{1}{3}$. Zero-field quantum Hall effect phases ($\nu = \pm 1$, where $\sigma^{xy} = \nu e^2/h$) occur if $|M/t_2| < 3\sqrt{3}\,|\sin\phi|$. This figure assumes that t_2 is positive; if it is negative, ν changes sign. At the phase boundaries separating the anomalous and normal ($\nu = 0$) semiconductor phases, the low-energy excitations of the model simulate undoubled massless chiral relativistic fermions.

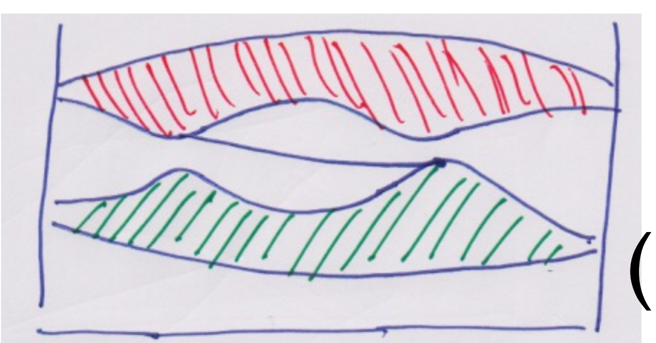


Broken inversion

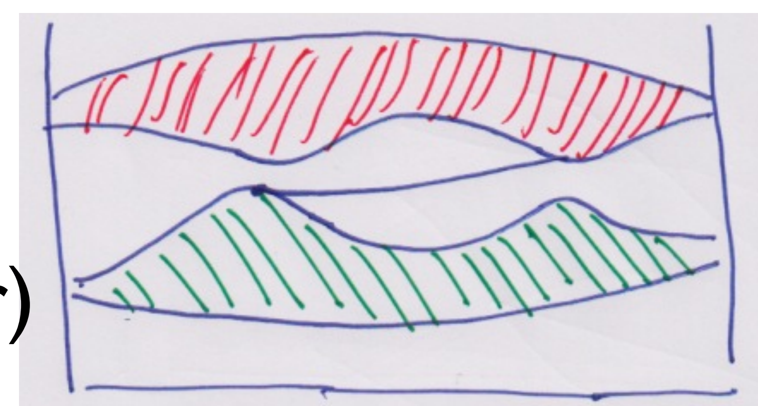


- gapless graphene "zig-zag" edge modes





Broken time-reversal (Chern insulator)



- Because of its simplicity, this model has been very useful in a number of contexts:

- It provides a simple model of a Berry-phase-driven quantum anomalous Hall effect when the Fermi level is in the gap.

- When the Fermi level is in one of the bands, it provides a simple model for a 2D unquantized anomalous Hall effect, where in 2D the Hall conductance is the total Berry phase for moving a quasiparticle clockwise around each branch of the Fermi surface. This result proved very useful for the more interesting generalization showing that the 3D anomalous Hall effect was a geometrical Fermi-surface Berry-curvature property (FDMH, Phys. Rev.Lett. 93, 206602 (2004)

Photonic "Chern insulator bands"

- Photons are neutral bosons, not charged fermions, so they will not exhibit a QHE
- but photonic bands with non-vanishing Chern number can have protected edgemodes if they are in a bulk photonic gap!
- My student Sri Raghu found a photonic bandstructure based on the graphene model that explicitly demonstrated this.
- Later, a (using a more robust bandstucture design), protected edge transport of microwaves was experimentally confirmed at MIT

Analogs of quantum Hall edge states in photonic crystals

Haldane and Raghu, Phys. Rev. Lett.100, 013904 (2008)

- Predicted theoretically that using magnetooptic (time-reversal-breaking) materials, photonic analogs of electronic quantum Hall systems could be created where topologically-protected edge modes allow light to only travel along edges in one direction, with no possibility of backscattering at obstacles!

- Effect was experimentally confirmed recently at MIT (Wang et al., Nature 461, 775 (8 October 2009).

- Obvious potential for technological applications! (one-way loss-free waveguides)

microwaves go around obstacle!

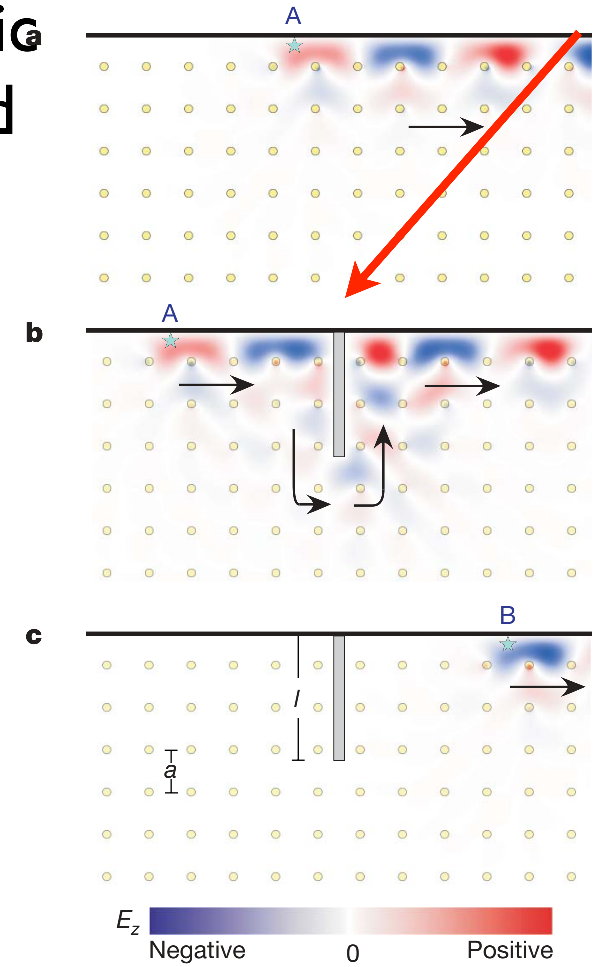
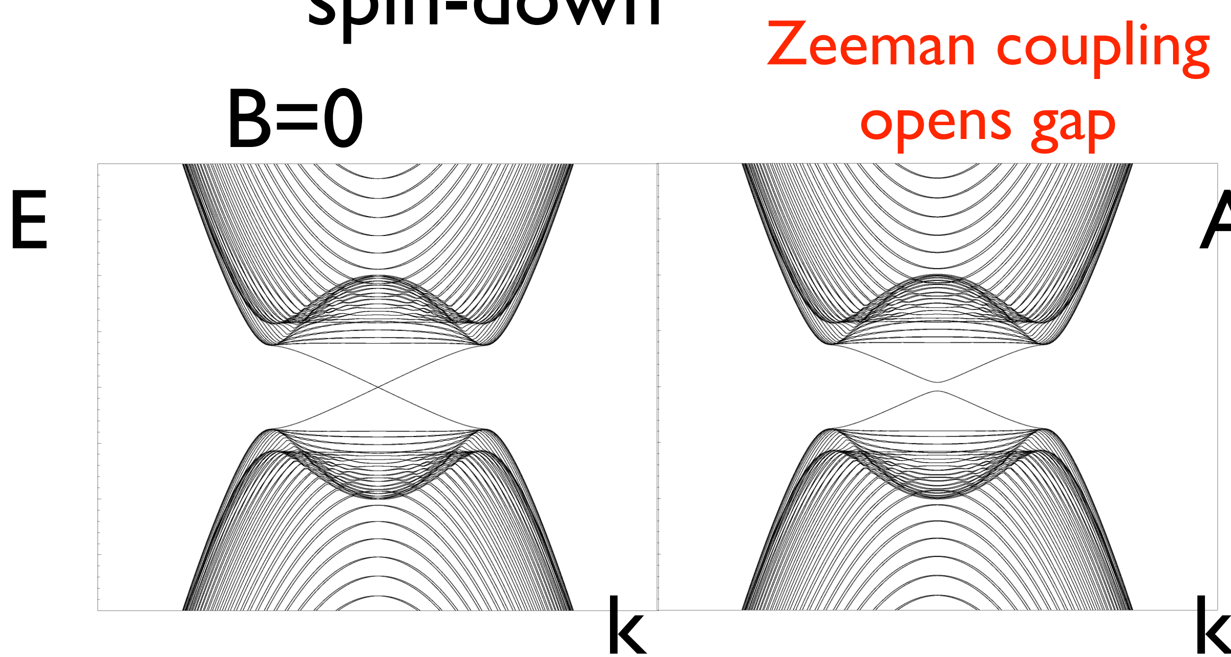


Figure 2 | Photonic CESs and effects of a large scatterer. a, CES field distribution (E_z) at 4.5 GHz in the absence of the scatterer, calculated from finite-element steady-state analysis (COMSOL Multiphysics). The feed

(from Wang et. al)

Kane and Mele 2005

- Two conjugate copies of the 1988 spinless graphene model, one for spin-up, other for spin-down



At edge, spin-up moves one way, spin-down the other way

If the 2D plane is a plane of mirror symmetry, spin-orbit coupling preserves the two kinds of spin. Occupied spin-up band has chern number +1, occupied spin-down band has chern-number -1.

- This looks "trivial", but Kane and Mele found that the gapless "helical" edge states **were still there** (!) when Rashba spin-orbit coupling that mixed spin-up and spin-down was added.
- They found a new "Z2" topological invariant of 2D bands with time-reversal symmetry that takes two values, +1 or -1. The invariant derives from **Kramers degeneracy** of fermions with time-reversal symmetry.
- This launched the new "topological insulator" revolution when an experimental realization was demonstrated by Molenkamp in a much improved, physically-realizable model designed by Andrei Bernevig, Taylor Hughes and Shoucheng Zhang

I'll leave the "modern" TI story for my co-medallists to tell, but I'll comment on the things that surprised me the most .....

- The 3D effect, first formally discovered by Joel Moore (whose undergraduate Senior Thesis I supervised years ago) and Leon Balents, and independently by Rahul Roy.

This really become clear after Charlie Kane and his then student Liang Fu found the beautiful formula that allows "Topological or not" to be determined by inspection of inversion-symmetric bulk 3D band structures

and

- The beautiful reformulation of the 3D TI as 3+1d "axion electrodynamics" by Xiao-Lin Qi, Taylor Hughes and Shoucheng Zhang:

this parallels the "theta term" of the 1+1d spin chain,based on the second Chern invariant as opposed to the first because the spatial dimension increased by 2!

- Finally, recent work by a number of authors on “flat band” fractional Chern insulators, including my 1988 honeycomb model has shown that they support Laughlin-like states on a lattice, making a start to the fractional topological insulator story!

The moral of this long story: suggests <u>three</u> distinct ingredients for success.

- Profound, correct, but perhaps opaque formal topological results (Invariants, braid group, etc)
- Profound, simple and transparent "toy models" that can be explicitly treated (The honeycomb Chern Insulator, the Kitaev Majorana chain, etc)
- Understanding the real materials needed for "realistic" (but more complex) experimentally-achievable systems that can bring "toy model results" to life in the hands of experimentalist colleagues.